Relativistic Model Field Theory*

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A relativistic generalization of the Lee model is constructed and solved in the first sector. The solubility is achieved with an indefinite metric and a redefinition of antiparticle operators which amounts to a selection rule prohibiting pair creation. The renormalization of the V-particle results in three dressed states, one of which is a ghost. The properties of these states and their dependence on cutoff and coupling strength are discussed. The effects of various modifications of the interaction are analyzed. Thirdly, the N-θ scattering problem is solved exactly.

I. INTRODUCTION

THIS paper presents another in a growing number of model field theories. As in all model theories, some of the physical features which are generally attributed to "real" theories are changed in order to make the model soluble in closed form (at least in principle).

The only real justification for working with a model which embodies patently nonphysical features solely for expediency is the fact that it can provide an exact solution while the real problem is insoluble. It is well known that some interesting qualitative insights into certain aspects of real theories can be gained through soluble models, but, on the other hand, one can never be certain that these insights will not prove to be false when (and if) the real theory is solved. The hope of the model builder is that this will not happen.

The axioms usually associated with a physical field theory are listed, for example, by Thirring.[1] (1) Lorentz invariance. (2) Positive energies. (3) Hilbert space and probabilistic interpretation. (4) Locality and causality. (5) Asymptotic conditions.

For our purposes, requirements (2) and (3) will be dropped; that is, we will construct a model which permits negative-energy solutions and possesses an indefinite metric. Our model will be a relativistic extension of the Lee model[2,3] in which three scalar bosons (called V, N, θ) interact via a simple scalar vertex.

A model similar to this has been examined by Gunther[4] who chose to treat the V and N particles as spinor fields and the θ as scalar. By doing this, he was able to avoid introducing an indefinite metric for the V and N (unless ghosts appeared), but once one allows an indefinite metric for the θ particle, there seems to be little extra conceptual difficulty in allowing it for all three. The simplification achieved by avoiding Dirac matrices is considerable.

Any fully relativistic theory must permit negative-energy solutions as a consequence of the quadratic energy-momentum relationship. Since we observe antiparticles in nature we know how to handle these negative-energy solutions in a real theory. But it is the presence of these antiparticles and the possibility of the unlimited virtual production of particle-antiparticle pairs which makes the real theory insoluble. In any perturbation or dispersion calculation, one is forced to include an infinite number of intermediate states. Gunther's[4] idea was to go back to the idea of a "completely empty vacuum" and eliminate antiparticles. The negative-energy solutions of the field equations are retained, but now they represent particles with negative energy in order to make the formalism consistent, states with negative norm. See Sec. II-A for more detail.

In this paper we will consider only the first sector, i.e., one V particle or an N–θ pair. We begin with a Lagrangian made up of an $\mathfrak{L}_0$ which is similar to the Lagrangian for three noninteracting charged scalar fields. We add to this a local $\lambda\varphi^3$ interaction. We are able to obtain exact solutions to the V-particle mass and charge renormalization and the N–θ scattering problem.

For the local interaction we find three renormalized V-particle states whereas only two bare V particles are postulated. The third root is shown to be a ghost state with a negative norm. This ghost exists for all values of the coupling constant and even with a cutoff. It is then demonstrated that both the ghost and the necessity for a cutoff can be eliminated by making the interaction nonlocal in a particular way. For this nonlocal interaction there is a wide range of coupling constant for which no ghosts appear.

Finally, the N–θ scattering problem is solved and the analytic properties of the S matrix exhibited.

II. NOTATION AND HAMILTONIAN

This model will consist of three non-Hermitian scalar fields which we call V, N, θ, after Lee. The only interaction in the theory is

$$V \rightleftarrows N+\theta \tag{2.1}$$

and the simplest local interaction Hamiltonian we can write for this is

$$H_I = g_0 \int d^3x \left(V^*(\mathbf{x}) N(\mathbf{x}) \phi(\mathbf{x}) + \text{H.c.}\right), \tag{2.2}$$

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[1] W. Thirring, Phys. Rev. **126**, 1209 (1962).
[2] T. D. Lee, Phys. Rev. **95**, 1327 (1954).
[3] G. Kallen and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).
[4] M. Gunther, Phys. Rev. **125**, 1061 (1962).

where we have chosen to write H_I in the Schrödinger picture. The free fields must satisfy the Klein-Gordon equation. This is ensured by writing the free Lagrangian in the standard charged scalar form[5]

$$\mathcal{L}_{0\theta}(\mathbf{x}) = :\partial_\lambda \phi^*(\mathbf{x})\partial^\lambda \phi(\mathbf{x}) - \mu^2 \phi^*(\mathbf{x})\phi(\mathbf{x}). \quad (2.3)$$

Equation (2.3) is written for the θ field and similar expressions are to be included for the V and N fields.

Now in a realistic field theory we would Fourier expand the field operators as follows:

$$\phi(x) = \frac{1}{(2\Omega)^{1/2}} \int_{k_0>0} \frac{d^3k}{\omega(\mathbf{k})} [a(\mathbf{k})e^{-ik\cdot x} + b^*(\mathbf{k})e^{+ik\cdot x}],$$

where $\Omega = (2\pi)^3$ and $a(\mathbf{k})$ destroys a particle and $b^*(\mathbf{k})$ creates an antiparticle. In our field theory we use the expansion

$$\phi(x) = \left(\frac{2}{\Omega}\right)^{1/2} \int d^4k\, \delta(k^2 - m^2) a(k) e^{-ik\cdot x}, \quad (2.4)$$

where the integral is over positive and negative energies. We require that $a_-(\mathbf{k})$ be a destruction operator, i.e., that

$$a_-(\mathbf{k})|0\rangle = 0. \quad (2.5)$$

The expansion (2.4) can be reduced to ($\epsilon = \pm$).

$$\phi(x,t) = \frac{1}{(2\Omega)^{1/2}} \int_{k_0>0} \frac{d^3k}{\omega(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_\epsilon a_\epsilon(\mathbf{k}) e^{-i\epsilon\omega t}. \quad (2.6)$$

But we want to ensure that the theory is still causal, that is, we want to retain the commutation relation:

$$[\pi(\mathbf{x}), \phi(\mathbf{x}')] = -i\delta^3(\mathbf{x} - \mathbf{x}'), \quad (2.7)$$

where

$$\pi(\mathbf{x}) = \frac{\partial}{\partial t} \phi^*(\mathbf{x},t)\Big|_{t=0}. \quad (2.8)$$

If (2.6) is substituted into (2.7), it is found that the commutation rules of the momentum-space operators must be

$$[a_\epsilon(\mathbf{k}), a_{\epsilon'}^*(\mathbf{k}')] = \epsilon\omega(\mathbf{k})\delta_{\epsilon\epsilon'}\delta^3(\mathbf{k} - \mathbf{k}'), \quad \omega(\mathbf{k}) = +(\mathbf{k}^2 + \mu^2)^{1/2}, \quad (2.9)$$

in order for (2.7) to be satisfied.

If operators of the form (2.6) are substituted into (2.2) it is easily seen that only combinations of the form $v_\epsilon^* n_\mu a_\nu$ and $n_\mu^* a_\nu^* v_\epsilon$ can occur.[6] In other words, we have satisfied the selection rule (2.1). But now we must live with the fact that the commutators (2.9) are abnormal. This abnormality results in states of negative norm. For example, a state containing a single negative energy θ particle of momentum $\mathbf{k}$ has the norm:

$$\langle\theta_-(\mathbf{k})|\theta_-(\mathbf{k})\rangle = \langle 0|a_-(\mathbf{k})a_-(\mathbf{k})|0\rangle = -\omega(\mathbf{k})\delta^3(0).$$

[5] See, e.g., S. S. Schweber, *Relativistic Quantum Field Theory* (Row-Peterson and Company, New York, 1961), pp. 195ff.

[6] The lower case letters, v, n, and a will denote momentum-space operators as opposed to the capital letters in coordinate space.

Methods for handling indefinite metrices have been discussed extensively by Nagy[7] and we will not go into them here. We will not introduce the so-called η formalism but will stick to the operators a_ϵ^*, a_ϵ and accept the fact that the eigenstates of the Hamiltonian do not span a Hilbert space. However, we will assume that they form a complete set and that no states of zero norm (dipole ghosts) can appear. As we will see later this will necessitate the introduction of cutoffs.

Starting with the Lagrangian (2.3) one derives the free Hamiltonian H_0 and when the expansions (2.6) are inserted it is easily shown that

$$H_{0\theta} = \int \frac{d^3k}{\omega(\mathbf{k})} \sum_\epsilon \omega(\mathbf{k}) a_\epsilon^*(\mathbf{k}) a_\epsilon(\mathbf{k}), \quad (2.10)$$

with similar expressions holding for the V and N. We begin then with the Hamiltonian[8]

$$H = H_{0\theta} + H_{0V} + H_{0N} + H_I,$$

where the momentum-space representation of H_I is found by combining (2.2) and (2.6).

III. PHYSICAL V PARTICLE

We may now use the Hamiltonian just described to determine the properties of the physical V-particle state. We first expand the physical state in terms of the bare states and specify that the V particle is in its rest frame:

$$|V(0)\rangle = \sum_\epsilon C(\epsilon)|V_\epsilon(0)\rangle_0 + \int \frac{d^3q}{\omega_N(\mathbf{q})\omega_\theta(\mathbf{q})} \sum_{\nu,\nu'} \Phi_{\nu\nu'}(q)|N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})\rangle_0. \quad (3.1)$$

This will be recognized as a generalization of the technique employed by Kallen and Pauli[3] in their original treatment of the Lee model. We now require

$$H|V(0)\rangle = m_V|V(0)\rangle, \quad (3.2)$$

where m_V can be either positive or negative.

The eigenvalue problem (3.2) leads to the following set of two equations:

$$m_V C(\epsilon') = \epsilon' m_0 C(\epsilon') + \frac{g_0}{(8\Omega)^{1/2} m_0} \int \frac{d^3q}{\omega_N \omega_\theta} \sum_{\nu,\nu'} \nu\nu' \Phi_{\nu\nu'}(q), \quad (3.3)$$

$$m_V \Phi_{\nu\nu'}(q) = (\nu\omega_N(\mathbf{q}) + \nu'\omega_\theta(\mathbf{q}))\Phi_{\nu\nu'}(q) + \frac{g_0}{(8\Omega)^{1/2}} \sum_{\epsilon'} \epsilon' C(\epsilon'). \quad (3.4)$$

To these equations we add the requirement that the

[7] K. L. Nagy, Nuovo Cimento Suppl. **17**, 92 (1960).

[8] In $H_0(V)$ we use the bare mass of the free V particle (m_0). We save the symbol m_v to denote the renormalized mass.

state be normalized:

$$\langle V(0)|V(0)\rangle = m_V\delta^3(0). \tag{3.5}$$

This leads to[9]

$$m_V = m_0\sum_\epsilon \epsilon C^2(\epsilon) + \int \frac{d^3q}{\omega_N\omega_\theta}\sum_{\nu,\nu'}\nu\nu'\Phi_{\nu\nu'}{}^2(q). \tag{3.6}$$

The mass renormalization is now obtained as follows: In Eq. (3.3) we notice that the integral term is independent of ϵ'. Using this fact we can write (3.3) for both $\epsilon'=\pm$ and add and subtract the two resulting equations. The subtraction leads to the equation

$$\sum_{\epsilon'} C(\epsilon') = \frac{m_V}{m_0}\sum_{\epsilon'}\epsilon' C(\epsilon'). \tag{3.7}$$

The addition leads to

$$m_V\sum_{\epsilon'}C(\epsilon') - m_0\sum_{\epsilon'}\epsilon' C(\epsilon') = \frac{2g_0}{(8\Omega)^{1/2}m_0}\int\frac{d^3q}{\omega_N\omega_\theta}\sum_{\nu,\nu'}\nu\nu'\Phi_{\nu\nu'}(q). \tag{3.8}$$

Now using (3.4) one solves for $\Phi_{\nu\nu'}(q)$ in terms of $\sum \epsilon' C(\epsilon')$ and inserts this and (3.7) into (3.8). The result is an equation from which the quantity $\sum\epsilon' C(\epsilon')$ cancels out leaving a relation which must be satisfied by the masses

$$m_V{}^2 - m_0{}^2 = 2\Omega^{-1}g_0{}^2F(m_V), \tag{3.9}$$

where

$$F(m_V) = \int\frac{d^3q}{8\omega_N\omega_\theta}\sum_{\nu,\nu'}\frac{\nu\nu'}{m_V-\nu\omega_N-\nu'\omega_\theta}. \tag{3.10}$$

The sum in (3.10) can be evaluated and the resulting integral

$$F(m_V) = m_V\int\frac{d^3q}{[m_V{}^2-(\omega_N+\omega_\theta)^2][m_V{}^2-(\omega_N-\omega_\theta)^2]} \tag{3.11}$$

is linearly divergent. The apparent ω^4 dependence in the denominator does not hold up because $\omega_N-\omega_\theta$ approaches a constant as q goes to infinity.

In order to solve (3.9) for m_V we must evaluate $F(m_V)$. For this purpose it is easiest to take $m_N=m_\theta=\mu$. Then $\omega_N-\omega_\theta=0$ for all q and $F(m_V)$ simplifies to[10]

$$F(m_V) = -\frac{4\pi}{m_V}\int_0^{q_m}\frac{q^2dq}{4\omega^2-m_V{}^2}. \tag{3.12}$$

[9] We have assumed the $C_\epsilon(\epsilon')$ and $\phi(q,\epsilon')$ to be real. It is not difficult to show that if m_v is real, then $\mathrm{Im}C_\epsilon(\epsilon')=0$ implies $\mathrm{Im}\phi_{\nu\nu'}(q,\epsilon)=0$. Therefore, in the following, whenever we use Eq. (2.16) we assume m_v real and adjust the phase so that $C(\epsilon)$ and ϕ are real.

[10] In using a step function cutoff we have not been careful about keeping Lorentz invariance. In the rest system of the V particle the volume in momentum space is spherical but in other frames it is distorted. For the purpose of this paper this is not an important consideration.

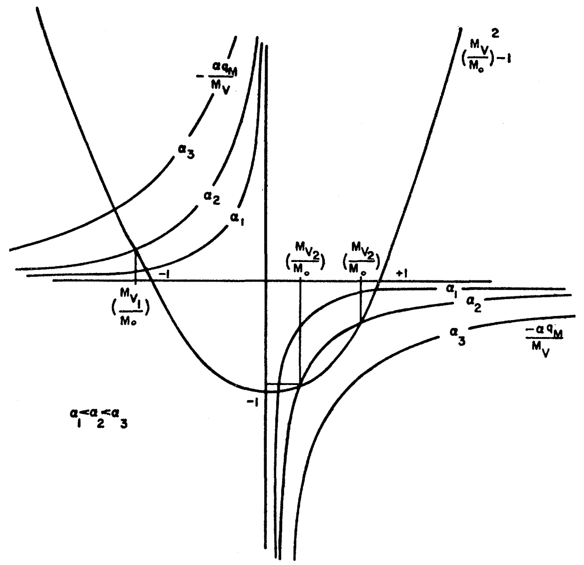

FIG. 1. Solution of Eq. (2.24) for the three "physical" V-particle states. q_m is the cutoff momentum and is assumed to be fixed while f_0 is varied.

(Note that $F(m_V)$ is negative.) This is a straightforward integral and the result is (assuming $q_m\gg 2\mu$)

$$F(m_V)\approx -\pi q_m/m_V. \tag{3.13}$$

Therefore,

$$m_V{}^2-m_0{}^2 = -\frac{2\pi}{\Omega}\frac{g_0{}^2g_m}{m_V} = -\frac{f_0{}^2}{\pi}m_0{}^2\frac{q_m}{m_V}, \tag{3.14}$$

where

$$f_0{}^2 = (1/m_0{}^2)(g_0{}^2/4\pi). \tag{3.15}$$

Equation (3.14) is a cubic and the roots can be located approximately by graphical means as shown in Fig. 1. There is always one real root with negative mass {and therefore negative norm [Eq. (3.5)]} and this root is always below $-m_0$. The other two roots can be real or complex depending on the size of the coupling constant.

The appearance of three bound states where we would only expect two (i.e., the renormalized v_+ and v_-) is an interesting feature of this model. The most important consequence of this extra root is the fact that the free theory cannot be obtained as the uniform limit of the interacting theory as $f_0\to 0$. However, in a certain sense the extra state does "disappear" in this limit as one can see by examining the charge renormalization, which we now consider.

We define the charge renormalization Γ by referring to Eq. (3.4). The effective coupling constant is evidently $g=g_0\Gamma$, where

$$\Gamma = \sum_\epsilon \epsilon C(\epsilon). \tag{3.16}$$

The sum in (3.16) is evaluated by combining (3.3), (3.4), and (3.6) and the result is (for real m_V)

$$\Gamma^2=\left[1+\frac{f_0^2}{4\pi}\frac{m_0^2}{m_V}\int_0^{q_m}\frac{q^2dq}{\omega^2}\sum_{\nu,\nu'}\frac{\nu\nu'}{(m_V-\nu\omega-\nu'\omega)^2}\right]^{-1}. \quad (3.17)$$

The summation and integral can be worked out explicitly, and if we assume that the linearly divergent term dominates, we find

$$\Gamma^2=\left[1-\frac{8f_0^2}{\pi}\frac{m_0^2}{m_V^3}\int_0^{q_m}\frac{q^4dq}{(m_V^2-4\omega^2)^2}\right]^{-1}.$$

The divergent part of the integral is simply $q_m/4$ and we have

$$\Gamma^2=\left[1-\frac{f_0^2}{2\pi}\frac{m_0^2q_m}{m_V^3}\right]^{-1}. \quad (3.18)$$

Using (3.14) to substitute for $m_0^2q_m$ and letting $\alpha=f_0^2\pi^{-1}$, we get

$$\Gamma^2(m_V)=\frac{2m_V^2}{3m_V^2-m_0^2}. \quad (3.19)$$

Therefore, for each root m_V, there is a different value of the charge renormalization, and Γ^2 is real and positive as long as m_V is not in the range $0\leq m_V\leq m_0/\sqrt{3}$. If Γ^2 is negative for some m_V, then this root is a ghost state in the sense defined by Kallen and Pauli.[11]

We are now able to return to the three roots shown in Fig. 1 and discuss their properties. The root m_{V_1} has negative mass and therefore negative norm. In the limit $\alpha\rightarrow 0$ it approaches $-m_0$ and from (3.19) we see that $\Gamma(m_{V_1})$ is real and less than 1; the latter showing that the effective coupling of this state is reduced by the presence of the interaction.

The roots m_{V_2} and m_{V_3} approach 0 and $+m_0$, respectively, as $\alpha\rightarrow 0$ and both have positive norm according to Eq. (3.5). An inspection of the hyperbola $-\alpha q_m m_V^{-1}$ shows that the root m_{V_2} is always greater than αq_m (as long as m_{V_2} is real) and the difference $m_{V_2}-\alpha q_m$ looks to be of second order in α. These observations can be verified by solving the cubic equation (3.14) analytically. We also note that as α is increased a value will be reached at which the hyperbola and parabola are just tangent at a single point. This point marks the transition of m_{V_2} and m_{V_3} from real to complex, and it can be shown that at this "critical point" (the dipole ghost solution)

$$m_{V\mathrm{crit}}=\tfrac{3}{2}\alpha q_m=m_0/\sqrt{3}. \quad (3.20)$$

Therefore, if m_{V_3} is real, it must be greater than $m_0/\sqrt{3}$ and is not a ghost. But $\Gamma(m_{V_3})$ is then real and *greater* than one. So the effective coupling constant of this root is enhanced by the interaction. This is an effect which should not occur in a realistic field theory.

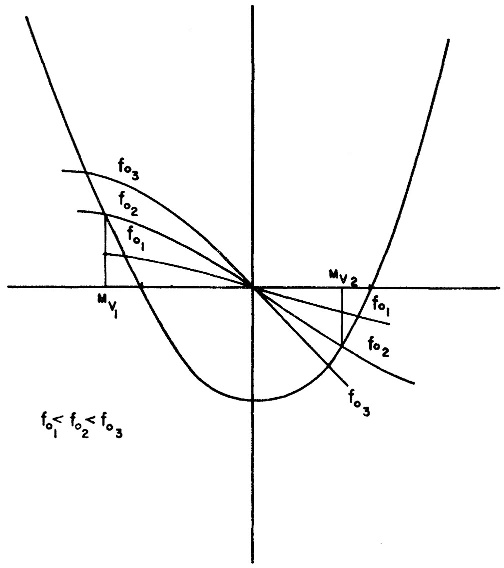

FIG. 2. Solution for model with $\nu\nu'=1$. There are two physical V particles and the renormalization is finite.

Now consider m_{V_2}. It is always a ghost state and in the limit $\alpha\rightarrow 0$ we find $\Gamma^3(m_{V_2})\rightarrow 0$. It is in this sense that we can say that the root m_{V_2} disappears in the no coupling limit. Its wave function is renormalized to zero by $\Gamma(m_{V_2})$ in this limit.[12]

Finally, we examine the behavior of the roots as α becomes large (strong coupling). Nothing special happens to m_{V_1} as it simply moves farther to the left. On the other hand, m_{V_2} and m_{V_3} become complex. For $\alpha q_m \gtrsim m_0$ we find that $m_{V_2}=m_{V_3}{}^*$ and

$$m_{V_2}\approx\frac{1}{2}\left(\frac{3\beta^2+1}{3\beta}\right)+\frac{i\sqrt{3}}{2}\left(\frac{3\beta^2-1}{3\beta}\right), \quad (3.21)$$

where

$$\beta=(\alpha q_m/m_0)^{1/3}.$$

For completeness we also give m_{V_1} in this limit:

$$m_{V_1}\approx-[(3\beta+1)/3\beta]. \quad (3.22)$$

Note that $m_{V_2}+m_{V_3}=-m_{V_1}$. This is a general result independent of the coupling strength and comes from the fact that the quadratic term in the cubic equation (3.14) is missing. It is also amusing to note that in the limit of large coupling, the three roots form the vertices of an equilateral triangle in the energy plane.

In the preceding discussion we have never specified the location of the threshold of the physical N–θ spectrum, except for the tacit assumption that $m_0<2\mu$. We now notice that all of our results are independent of

[11] If $\Gamma^2<0$ for some m_v our assumption in footnote 9 is not valid. States of this type must be treated in the way Kallen and Paulin treated their ghost state, i.e., an *additional* indefinite metric must be introduced.

[12] In the more general case, where $m_N\neq m_\theta$ and we let $m_N-m_\theta=\Delta$, the hyperbola is asymptotic to the line $m_V=\Delta$ rather than $m_v=0$. In this case if α is decreased to zero $m_{v2}\rightarrow +Q$. The result $\Gamma(m_{v2})\rightarrow 0$ still holds.

μ as long as $q_m \gg 2\mu$, which can always be satisfied since both α and q_m are adjustable. But we have kept only the leading terms in the integrals of Eqs. (3.12) and (3.17). The convergent parts do depend on μ and by examining these we find that if $m_V > 2\mu$ (or $m_V < -2\mu$), the integrals have small convergent imaginary parts. These solutions are then to be interpreted as N–θ scattering resonances with the position of the resonance determined primarily by αq_m and the width determined primarily by α and μ. In general, we would expect the widths of m_{V_2} and m_{V_3} to be considerably greater than that of m_{V_1}, since the former leave the real axis even before the threshold is reached.

Before closing this section, we consider briefly three modifications of the Hamiltonian which lead to different sets of roots. We can write the interaction in momentum space by combining Eq. (2.2) with the expansion of (2.6) of the field operators. The result is

$$H_I = \frac{g_0}{(8\Omega)^{1/2}} \int \frac{d^3p}{p_0} \int \frac{d^3q}{q_0} \int \frac{d^3k}{k_0} \delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \times \sum_{\epsilon,\mu,\nu} v_\epsilon^*(\mathbf{p}) n_\mu(\mathbf{q}) a_\nu(\mathbf{k}) + \text{Hermitian conjugate}. \quad (3.23)$$

We know that H_I is local from (2.2). It can be made nonlocal by dropping terms from the summation over energy indices and we wish to point out three of the ways in which this can be done.

(1) Keep only terms in which $\mu\nu = +1$. This eliminates the $N_+\theta_-$ and $N_-\theta_+$ states and removes the center cut in the energy plane.

(2) Keep only terms in which $\epsilon = \mu\nu$. This is a "conservation of norm" selection rule and it eliminates mixing of v_+ and v_- states.

(3) Keep only terms in which $\epsilon = \mu = \nu$. This completely uncouples the positive and negative energy regimes and is equivalent to two Lee models side by side.

The most interesting of these three modifications is the first. It leads to the set of two solutions illustrated in Fig. 2 and has the added virtue of being convergent. The masses m_{V_1} and m_{V_2} are the roots of the equation

$$m_V^2 - m_0^2 = \frac{f_0^2 m_0^2 \mu}{2m_V}\left[\left(1-\frac{m_V^2}{4\mu^2}\right)^{1/2} - 1\right] \quad (3.24)$$

and the charge renormalization is given by

$$\Gamma^2 = \left\{1 + \frac{f_0^2 \mu m_0^2}{4m_V^3}\left[\left(1-\frac{m_V^2}{4\mu^2}\right)^{-1/2} - 1\right]\right\}^{-1}, \quad (3.25)$$

which, with the help of (3.24) can be reduced to

$$\Gamma^2 = \left(1-\frac{m_V^2}{4\mu^2}\right)^{1/2}\left[\left(1-\frac{m_V^2}{4\mu^2}\right)^{1/2} - \frac{1}{2}\left(1-\frac{m_0^2}{m_V^2}\right)\right]^{-1}. \quad (3.26)$$

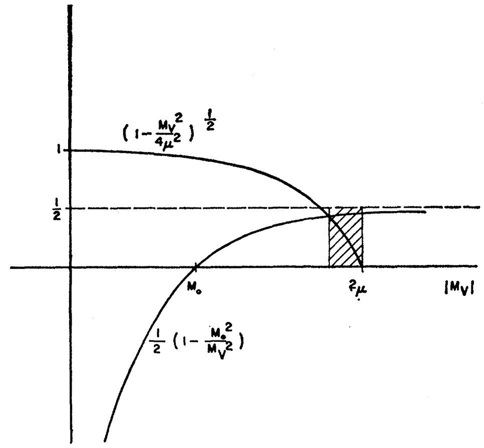

FIG. 3. Graph of Eq. (2.36) showing values of $|m_V|$ (shaded region) for which $\Gamma^2(m_V) < 0$.

In order to find the value of m_V at which Γ^2 becomes negative we plot in Fig. 3 the two terms in the square brackets of (3.26). In the shaded region Γ^2 is negative. Since one term depends only on the threshold $4\mu^2$ and the other only on m_0, these two parameters can be varied independently as long as $m_0 < 2\mu$. Also note that (3.26) contains only m_V^2 which allows us to plot just the absolute value of m_V.

In this modified model the extra root does not appear and both the v_+ and v_- particles are renormalized to lower masses. The coupling constant of m_{V_1} is increased and that of m_{V_2} decreased by the interaction.

The removal of the $N_+\theta_-$ and $N_-\theta_+$ states has therefore changed a divergent model to a convergent one. An examination of the second of the models listed above confirms the expectation that these N–θ states are responsible for the divergence. Model 2 leads to a finite renormalization for the v_+ which couples to the $N_+\theta_+$ and $N_-\theta_-$ and an infinite renormalization for the v_- which couples to the $N_+\theta_-$ and $N_-\theta_+$. Finally, model 3 gives infinite renormalizations for both V particles because only one of the $N_+\theta_+$ or $N_-\theta_-$ couples to each of the two energy states of the V.

IV. N–θ SCATTERING

By applying the above techniques to the N–θ scattering problem we can obtain the exact physical eigenstates and from these the S matrix. The physical N–θ state is expanded in terms of the bare states as follows:

$$|N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})\rangle = |N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})\rangle_0 + \int \frac{d^3q'}{\omega_N'\omega_\theta'} \sum_{\mu,\mu'} \alpha_{\nu\nu',\mu\mu'}{}^{\text{in}}(\mathbf{q},\mathbf{q}') |N_\mu(\mathbf{q}')\theta_{\mu'}(-\mathbf{q}')\rangle_0 + \sum_\epsilon \beta_{\nu\nu'\epsilon}{}^{\text{in}}(\mathbf{q}) |V_\epsilon(0)\rangle_0, \quad (4.1)$$

where we have already specified the center-of-mass system and where "in" on the physical state has the

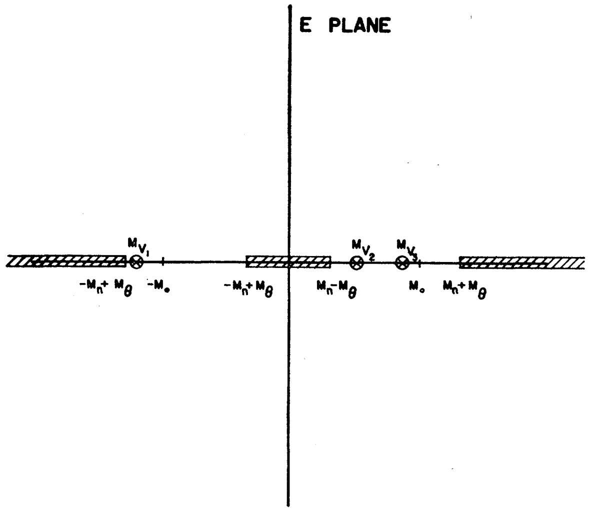

FIG. 4. Singularities of N-θ scattering S matrix. We have taken $m_N \neq m_\theta$ to exhibit the center cut.

usual meaning of incoming plane waves at $t=-\infty$ and only outgoing scattered waves at $t=+\infty$.

Solving the eigenvalue problem,

$$H|N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})\rangle = E_{\nu\nu'}(\mathbf{q})|N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})\rangle, \quad (4.2)$$

we get

$$\alpha_{\nu\nu'\mu\mu'}{}^{\text{in}}(\mathbf{q},\mathbf{q}') = \frac{g_0}{(8\Omega)^{1/2}} \sum_\epsilon \epsilon\beta_{\nu\nu'\epsilon}{}^{\text{in}}(\mathbf{q}) \times (E_{\nu\nu'}(\mathbf{q}) - E_{\mu\mu'}(\mathbf{q}') + i\tau)^{-1}, \quad (4.3)$$

$$\beta_{\nu\nu'\epsilon}{}^{\text{in}}(\mathbf{q}) = \frac{g_0\nu\nu'}{(8\Omega)^{1/2}m_0} \times \left[\frac{E_{\nu\nu'}(\mathbf{q}) + \epsilon m_0}{(E_{\nu\nu}(\mathbf{q})+i\tau)^2 - m_0^2 - (2g_0^2/\Omega)F(E+i\tau)}\right], \quad (4.4)$$

where

$$E_{\nu\nu'}(\mathbf{q}) = \nu\omega_N(\mathbf{q}) + \nu'\omega_\theta(\mathbf{q}), \quad (4.5)$$

and $F(E+i\tau)$ is given by (3.10) with the substitution $m_V \to E_{\nu\nu}+i\tau$.

The S matrix is calculated from

$$S_{fi} = {}_{\text{out}}\langle N_\nu(\mathbf{q})\theta_{\nu'}(-\mathbf{q})|N_\mu(\mathbf{q}')\theta_{\mu'}(-\mathbf{q}')\rangle_{\text{in}} \quad (4.6)$$

and with the aid of (4.1), (4.3), and (4.4) this is found to be

$$S_{fi} = \omega_N(\mathbf{q})\omega_\theta(\mathbf{q})\nu\nu'\delta_{\nu\mu}\delta_{\nu'\mu'}\delta^3(\mathbf{q}-\mathbf{q}') - 2\pi i\delta^4 \times (p_f - p_i)\frac{g_0^2(4\Omega)^{-1}}{E_{\nu\nu}{}^2 - m_0^2 + (2g_0^2/\Omega)F(E+i\tau)}. \quad (4.7)$$

The denominator of the second term is of course the eigenvalue equation [cf. Eq. (3.14)] for the physical V-particle masses. Therefore, the S matrix has poles at each of these masses. It is also clear that the singular integral leads to the cuts shown in Fig. 4. *Note added in proof.* The cut structure of Fig. 4 is very suggestive of the cut structure obtained by Frautschi and Walecka [S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960)] for pion-nucleon scattering. However, the comparison of this model with theirs is not a fruitful one. Because of the indefinite metric in our model, the signs of the discontinuities across the cuts are changed and completely different results are obtained.

The three poles of the S matrix can be exhibited more clearly by expanding the second term in partial fractions. As long as E does not lie on any of the cuts we can write:

$$T_{fi} = \frac{g_0^2(4\Omega)^{-1}}{E^2 - m_0^2 + (\alpha q_m m_0^2/E)}. \quad (4.8)$$

If the denominator is written

$$E^2 - m_0^2 + \frac{\alpha q_m m_0^2}{E} = \frac{1}{E}(E-m_1)(E-m_2)(E-m_3),$$

then we have the relations

$$m_1+m_2+m_3=0,$$
$$m_1m_2+m_2m_3+m_1m_3 = -m_0^2, \quad (4.9)$$
$$m_1m_2m_3 = -\alpha q_m m_0^2.$$

By expanding (4.8) in partial fractions and using (4.9), one can show that

$$T_{fi} = \frac{1}{4\Omega}\sum_{i=1}^{3}\frac{g_0^2\Gamma^2(m_i)}{m_i^2}\frac{E}{E-m_i}. \quad (4.10)$$

The dimensionless renormalized coupling constant is, therefore,

$$g_i^2 = \frac{1}{4\Omega}\frac{g_0^2\Gamma^2(m_i)}{m_i^2},$$

where $\Gamma^2(m_i)$ is defined by (3.19).

V. PERTURBATION THEORY AND RENORMALIZATION

In view of the fact that we have an exactly soluble theory, it is interesting to compare the exact results with those which would be obtained with a standard application of perturbation theory. In particular, we will consider the physical V-particle propagator and its mass renormalization.

The inapplicability of standard renormalization techniques becomes evident at the outset. Ordinarily, the first step is to replace the m_0^2 term in H_0 by m_V^2 and then subtract δm^2 from H_I. But in doing this we implicitly assume that the renormalized (i.e., dressed) V particle is a solution of the Klein-Gordon equation:

$$(\Box + m_V^2)V(x) = 0. \quad (5.1)$$

Evidently such an assumption is in conflict with the

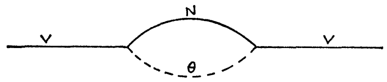

FIG. 5. Mass renormalization bubble diagram.

results of the previous sections, which gave three dressed V-particle states and an asymmetric mass shift.

However, suppose that we were to ignore the above objection and perform the usual mass renormalization anyway. We can transform H_I to the interaction picture and reduce the S matrix to a set of Feynman graphs as usual. Then we can calculate δm^2 to second order in g_0 by evaluating the bubble diagram of Fig. 5.

The rules for Feynman diagrams are very similar to those for charged scalar mesons except for the following two differences:

(1) There will be factors of $\epsilon=\pm 1$ which accompany external lines of positive and negative energies.
(2) If one evaluates the contractions between field operators which appear in the Wick[13] reduction it turns out that they are proportional to the *retarded* propagators rather than the Feynman propagators. This occurs because in the diagram both positive- and negative-energy particles move forward in time instead of the usual situation where antiparticles move backward in time.

In second-order perturbation theory, this behavior results in the lack of any mass renormalization diagrams for the N and θ particles and in the absence of vacuum fluctuation diagrams. This, of course, is the intended behavior, and it is quite instructive to see just how it is tied up with the use of retarded instead of Feynman propagators.

Using these rules one can write out the amplitude of Fig. 5 and after some algebraic manipulation it is found that δm^2 must satisfy Eq. (3.9), i.e.,

$$\delta m^2 = -\alpha q_m m_0^2/m_V. \qquad (3.9a)$$

So we can still get the correct answer if we interpret (3.9a) as a cubic equation for m_V.

This equivalence of the perturbation and exact solutions is a consequence of the extreme simplicity of our model. In a slightly more complicated model in which diagrams such as Fig. 6 can occur, one would not expect this equivalence.

We would like to be able to conclude this section by writing the Hamiltonian in completely renormalized form. However, as yet we have not been able to accomplish this, because of an inability to define a satisfactory renormalized V-particle operator and the commutation rules it must satisfy. If such an operator (actually a set of three operators) can be constructed it will certainly be nonlocal in character. It is possible that such operators can be constructed with the "dressing transformation" techniques of Greenberg and Schweber,[14] but we have no answer to this at this time.

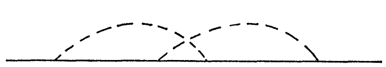

FIG. 6. Two meson renormalizations which cannot occur in our simple model.

VI. CONCLUSION

We have analyzed an exactly soluble relativistic model field theory. The model was made soluble in closed form by the elimination of antiparticles, which in turn required the use of an indefinite metric and the introduction of negative-energy particles. Except for one minor deviation (the cutoff procedure), relativistic invariance has been maintained throughout.

The most interesting feature of this model is the appearance of an extra root of the mass renormalization equation. For small coupling this extra state has very small mass and is a ghost state. For large coupling it joins with another root to form a complex doublet with equal absolute masses. In the limit of very strong coupling the three roots form a triplet of states all having the same absolute mass.

The mass renormalization is divergent. The divergence was shown to be removed if the interaction was constructed so as to forbid $N_+\theta_-$ and $N_-\theta_+$ intermediate states. It was also shown that these states are responsible for the extra root in the eigenvalue problem. This result provides a direct link between the divergence of the mass renormalization and the point nature of the interaction, which is not surprising. However, it is of some interest that the presence of the extra root is also tied up with locality.

We have discussed this model entirely from the view point of Lagrangian formalism. However, many of the features of this model are quite similar to those of a model constructed by Zachariasen,[15] based entirely on dispersion relations and unitarity. Although our model is not equivalent to Zachariasen's, it provides a good deal of insight into the connection between Lagrangian theories and S-matrix theories. We plan to discuss this connection in detail in a subsequent paper.

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